Length Contraction and Time Dilation, as Manifested in Special Relativity, Explained with Euclidean Geometry

Part 2

By Bob Duhamel

In my previous essay, I examined how observers in two airplanes, traveling at a constant speed, would see each other as they traveled together and on diverging and converging paths. I then showed that plotting time as a fourth dimension—combining the three dimensions of space into one dimension on the graph—implies that time dilation occurs when observers move through space, based on the implications of the two airplanes moving in space.

In this essay, we will look at Einstein's thought experiment, which examined how two observers would see lightning strike both ends of a train. He assumed there were two observers, one at the center of the moving train and the other beside the track. Just as the observer on the train passes the observer by the track, the observer by the track sees lightning strike both ends of the train simultaneously. Einstein postulated that the observer on the train would see the lightning strike the front of the train before the rear.

Curiously, Einstein explains why this happens incorrectly. He describes the observer on the train closing with the oncoming light at a combined speed that exceeds the speed of light. This violates the tenets of special relativity. He does this while explaining classical mechanics and relativity in the restricted sense and has yet to explain the principles of special relativity. Later, he shows that combining the velocities in that way conflicts with the principle that light travels at the same speed for all observers. However, he doesn't explicitly correct his original statement.

Let's start with the train stationary with the observers in their places, with the observer at the center of the train beside the observer by the track. This eliminates any relativistic effect.[1] To simplify the math, let's make the train as long as the distance light travels in two seconds. We will call this distance two light seconds

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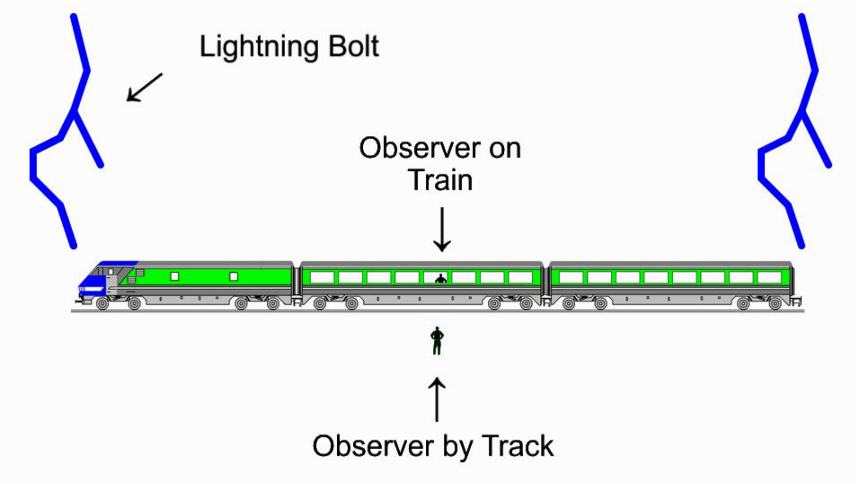
[1] You may hear that "relativistic speed" is required for time dilation and length contraction to occur. The effects of special relativity occur at any speed but are very small at the speeds we usually experience. You must go about 14 percent of the speed of light (about 42,000 kilometers per second) to get a length contraction or time dilation of one percent. The magnetic field surrounding a wire with an electrical current is due to the length contraction of the space between moving electrons. Within a two-millimeter diameter wire with one ampere of current, the electrons flow at only about 0.000025 meters per second. This causes an infinitesimally small length contraction. Yet, with approximately 2.5×10^{20} moving electrons per millimeter, the effect is significant.

or 2 Ls. Therefore, each observer is one light second (299,792,458 meters) from either end of the train.

Now, let's arrange for two bolts of lightning to strike the ends of the train simultaneously. Of course, each observer sees the lightning strike both ends of the train simultaneously.[2]



Two observers see lightning strike both ends of a train simultaneously.

Let's plot this on a time and distance graph where the X-axis (horizontal) represents time, and the Y-axis (vertical) represents distance.

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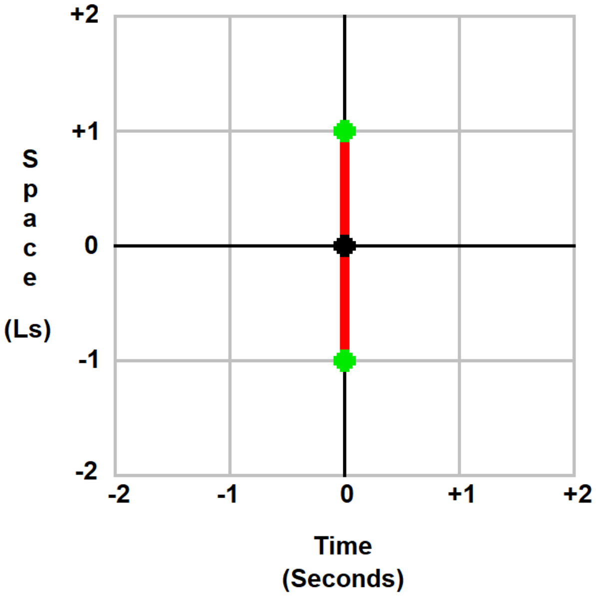
[2] We are *not* going to take into account the time it takes for the light to reach the observers. This has no effect on the outcome but only delays the time at which the observers see the lightning. Keep in mind that the effects of special relativity are not caused by propagation delay, as many non-physicists incorrectly explain. The delay between an event and when it is seen by an observer must added to the effects of special relativity to calculate when an observer will see an event.



A time and distance (space) graph with the train stationary

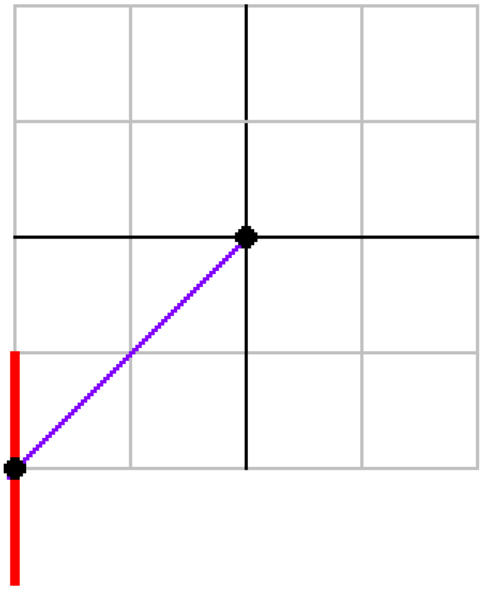
Each vertical line represents a time interval of one second, and each horizontal line represents a distance of one light second. Therefore, the train extends one line vertically above and below the center where the observers are placed.

The graph is centered on the train's center. The train (represented by the red line) extends one Ls (299,792,458 meters) in each direction vertically. This puts the rear of the train at -1 Ls and the front at +1 Ls in space. The black dot represents both observers; the observer by the track is next to the observer on the train, so the observers are at the same place (or nearly so) in space. The green dots represent the striking lightning bolts (we will use gray dots to represent those points in space when the lightning is not striking).

Note that even though the train and the observers are stationary, they are moving from left to right through time. The graph shows the moment the lightning strikes, which we will call T=0. The time before the lightning strikes is T-minus some seconds, and the time after the lightning strikes is T-plus some seconds. Therefore, one second before the lightning strikes is T-1 second, and one second after the lightning strikes is T+1 second.

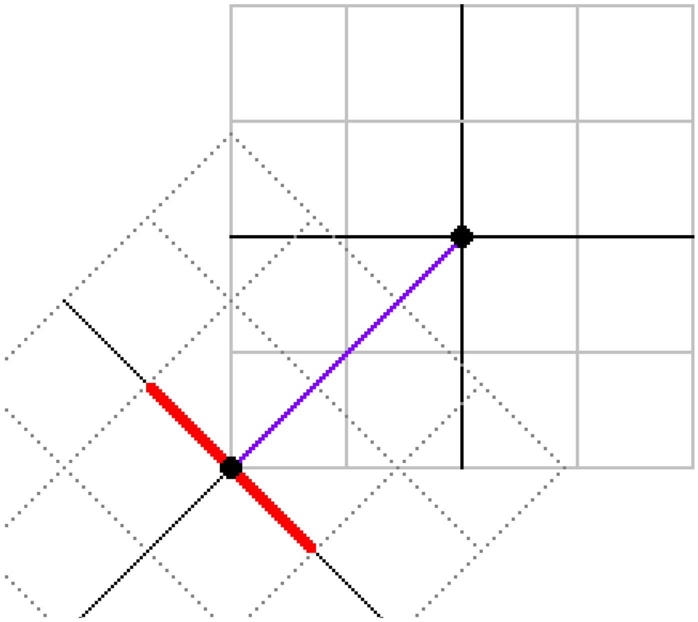
Now, let's roll the train backward some appropriate distance and start it rolling toward the observer by the track at 70.7 percent of the speed of light. This will move the train vertically, one light second for each second in time. As I showed in the previous essay, this will cause the train to move through the spacetime graph at a 45-degree angle to the time axis.

Without considering special relativity, we should plot the train as a line that is two light-seconds long and parallel to the timelines of the graph.



This diagram shows the train as it approaches the observer by the track moving at 70.7 percent of the speed of light.

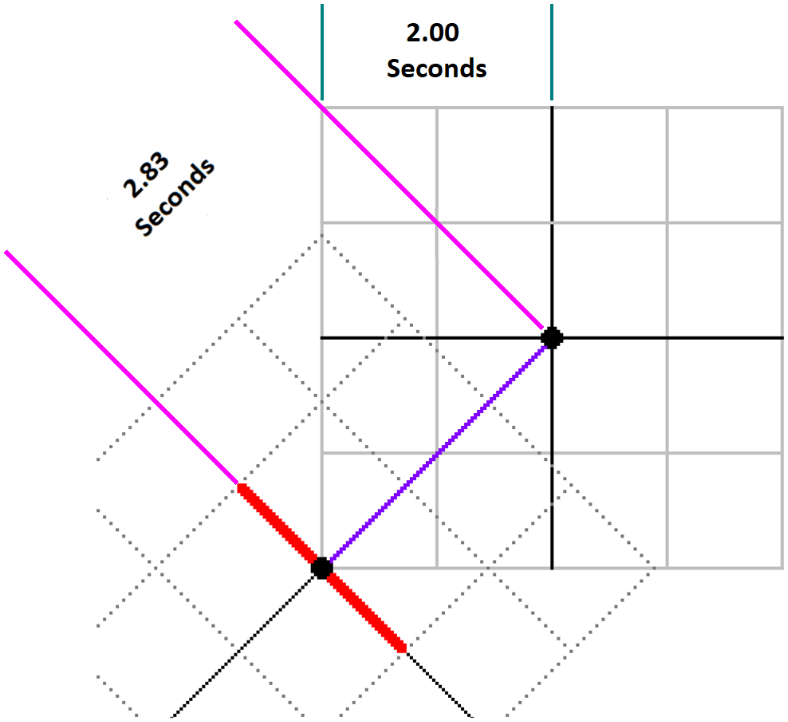
However, special relativity requires that no observer's frame of reference is preferred over another. As shown in the previous essay, when one observer moves through space as seen by another observer, the frame of reference of the moving observer is rotated relative to that of the stationary observer. Therefore, on our space vs. time graph, the frame of reference of the observer on the train is rotated 45 degrees to the frame of reference of the observer by the track.



The frames of reference of the two observers are rotated relative to each other.

This leads to two consequences of special relativity.

First, each observer sees the other observer moving slower through time than him or herself (refer to the first essay for a detailed explanation). At the moment shown above, this is easiest to see along the course through time of the observer on the train (the purple line in the graph).

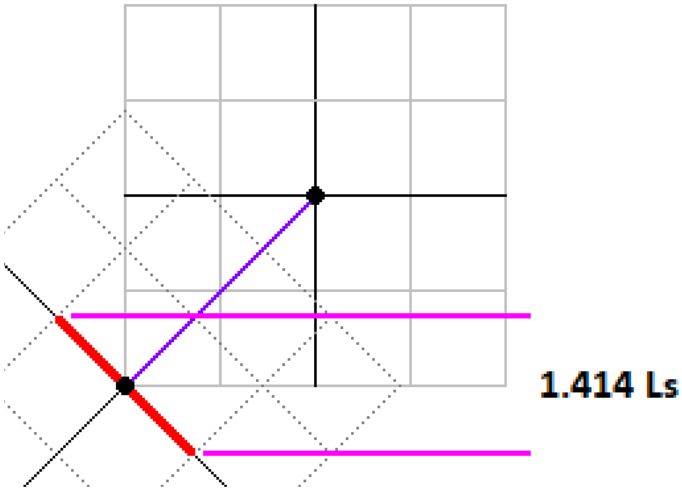


The train and its observer take 2.83 seconds to travel through two seconds of the frame of reference of the observer by the track.

Notice that it takes the observer on the train 2.83 seconds to go through two seconds in the frame of reference of the observer by the track.

Careful examination of the graph will show that it likewise takes the observer by the track 2.83 seconds to go through two seconds of time in the frame of reference of the observer on the train. This will be easier to see later when we look at the scenario as seen by the observer on the train. Each observer sees the other observer moving through time slower than him or herself.

Second, the train only extends 1.414 light seconds through space in the frame of reference of the observer by the track.

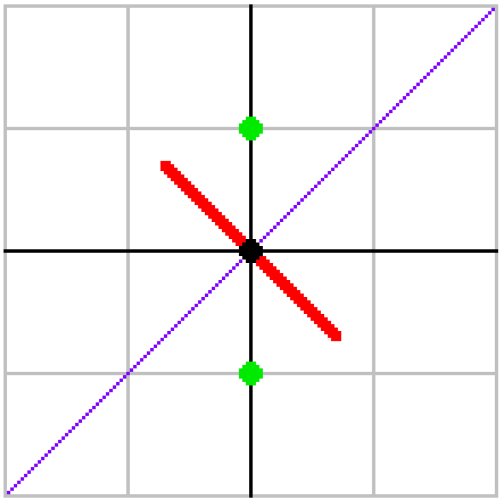


The train only extends 1.414 light seconds in the frame of reference of the observer by the track. Therefore, the observer by the track sees the train as only 1.414 Ls long instead of the 2 Ls when stationary.

Let's say you have a stick that is one meter long. If you look at that stick angled to you by 45 degrees, it will appear foreshortened to 0.707 meters. The stationary observer sees the train foreshortened in spacetime. Relativistic length contraction is a similar foreshortening of moving objects as viewed from a stationary frame of reference.

Also, notice that the rotation changes where the ends of the train end up in time; the rear of the train moves forward in time, and the front of the train moves backward in time. This shows that length contraction and time dilation are two sides of the same coin.

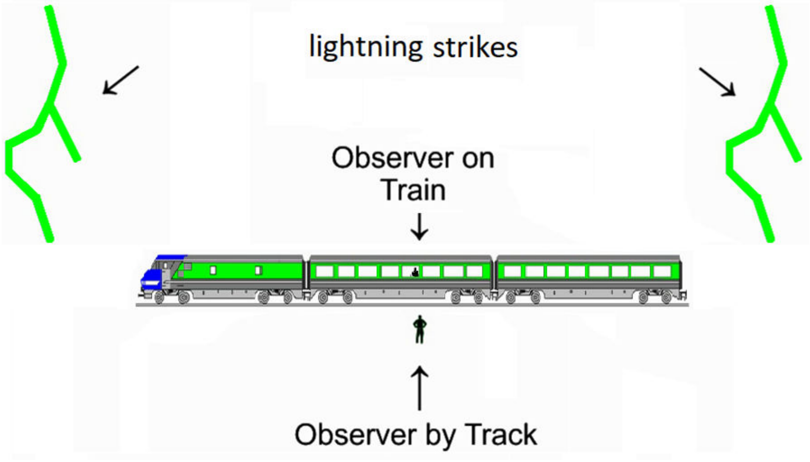
Let's see what happens when the train reaches the point where the observer on the train meets the observer by the track. This is when the lightning strikes in the frame of reference of the observer by the track. To avoid confusion, let's remove the space vs. time grid for the observer on the train.



At T=0, the observer on the train meets the observer by the track.

At this time, we encounter two problems:

First, the train is length contracted in the frame of reference of the observer by the track, so the lightning bolts are too far apart to hit the train simultaneously.



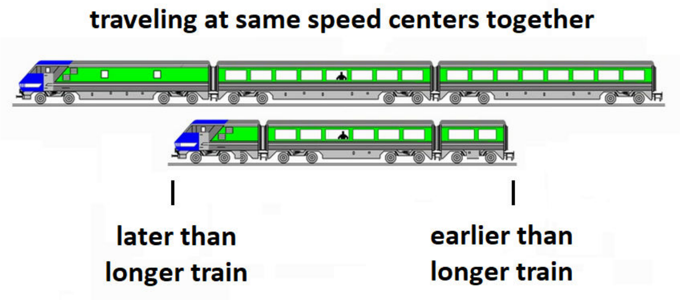
The lightning strikes are too far apart for the length-contracted train.

Our scenario requires the observer by the track to see lightning strike the moving train simultaneously at both ends. For the scenario to work with the moving length-contracted train, we must arrange for the lightning bolts to be placed closer together to a distance of 1.414 Ls.

Second, the front of the train is earlier in time by 0.707 seconds, and the rear is later by 0.707 seconds; it appears the different parts of the train are at different times. We must interpret the graph carefully to see what this means. Let's take a practical look at the train rolling along the track to interpret the diagram. This will show that nothing bewildering is happening with time.

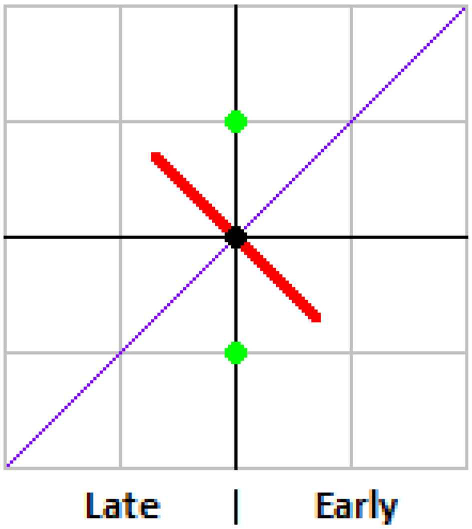
Let's say we have a one-kilometer-long train passing through a station at 60 kilometers per hour. The front of the train will pass through the station one minute before the rear of the train. Let's do this again with a train that is only 500 meters long. The rear of the shorter train will pass through the station only 30 seconds after the front. We changed the time that the rear of the train passes through the station by shortening the train.

Let's put our shorter train side-by-side with the longer train with the centers together.



Two trains of different lengths traveling at the same speed with their centers together

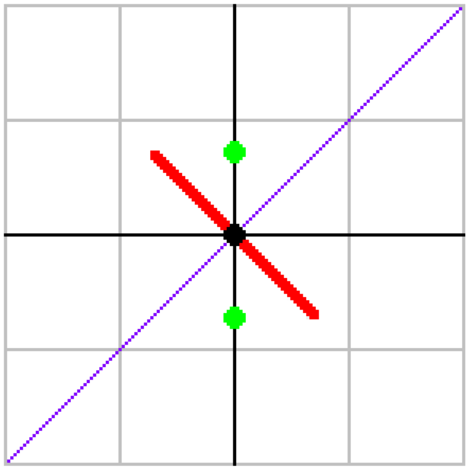
The front of the shorter train passes through the station later than the front of the longer train. Likewise, the rear of the shorter train passes through the station earlier than the rear of the longer train; shortening the train causes the front to be late and the rear to be early. The graph says the front is behind in time and the rear is ahead in time. The graph simply tells us that the length-contracted train is shorter. Since we are centering our graph on the center of the train, the front of the length-contracted train encounters points in space later than the front of the full-length train, and the rear of the length-contracted train encounters points in space earlier than the rear of the full-length train.



The front of the train is rotated back on the time axis, making it late, while the rear is rotated forward on the time axis, making it early.

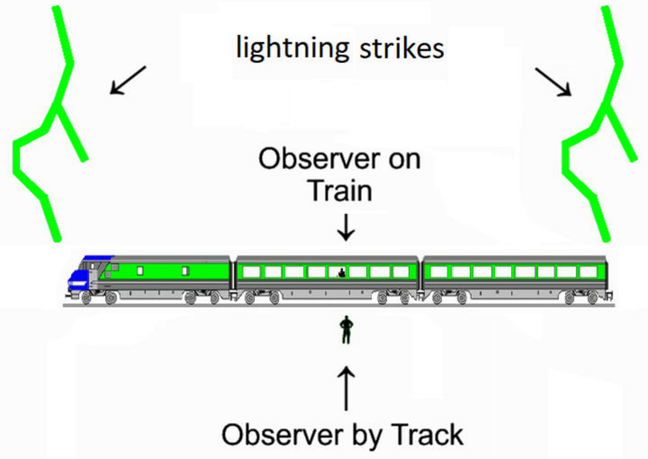
The observer by the track will not need a time machine to see the ends of the train. He or she will simply see a length-contracted train with a length of 1.414 light seconds moving along the track. Due to the shorter length, the front of the train will arrive later than the front of the full-length train, and the rear of the train will arrive earlier than the rear of the full-length train.

Now that we are armed with a correct interpretation of the graph, let's arrange for our lightning bolts to strike both ends of the train simultaneously at the correct distance.



Here, the lightning bolts have been moved to a distance of 1.414 Ls apart.

Now that we have moved the lightning bolts closer together to a distance of 1.414 light seconds, they can strike both ends of the length-contracted train simultaneously. The graph now depicts the train extending 1.414 light seconds in the frame of reference of the observer by the track (the ends of the train are even horizontally with the points in space where the lightning strikes). Though the graph doesn't show the ends of the train aligning with the lightning strikes on the time axis, we know that this just means that the ends of the length-contracted train meet points in space at different times than the ends of the full-length train. The observer by the track simply sees lightning strike both ends of the train simultaneously.



Our setup has lightning strike both ends of the train simultaneously as the observers meet.

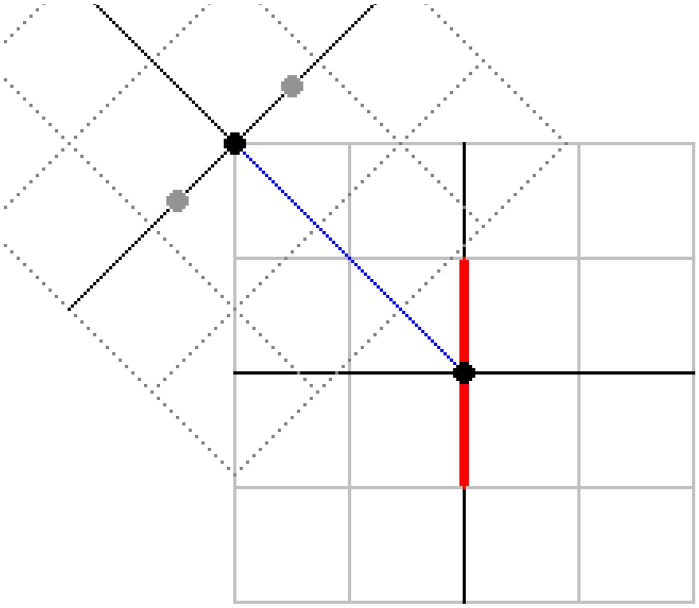
Now, our thought experiment is set up. We designed the experiment so that lightning strikes both ends of the moving, length-contracted train simultaneously, as seen by the observer by the track. Next, we need to determine what the observer on the train sees.

From the frame of reference of the observer on the train

We first need to look at a few points in time to see how space and time line up in this case.

First, the observer on the train sees him or herself as stationary. To the observer on the train, everything outside the train appears to be moving in the opposite direction to that which the train was moving as seen from the frame of reference of the observer by the track. Being stationary in space—

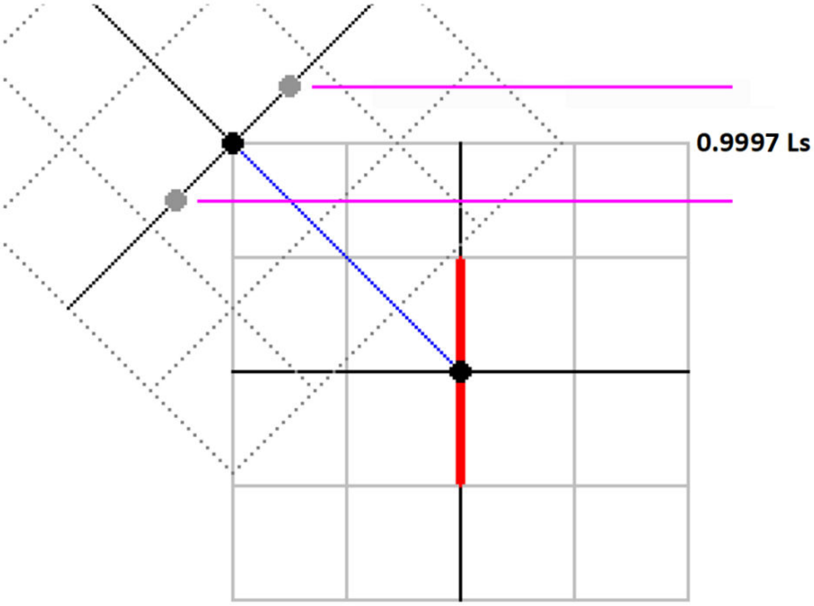
in his or her frame of reference—the observer on the train now appears on the space=0 line of the space axis of the graph (the black horizontal line at the center of the graph). Regardless of the time, the observer on the train will be at the center of the Y-axis of the graph. Since the observer on the train sees the train as not moving, the train is not length-contracted; it extends a full light second above and below the center line. The observer on the train sees the observer by the track as approaching from ahead of the train, along with the points in space where the lightning strikes. This puts the observer by the track and the points in space where the lightning strikes to the upper left on the graph.



Here is the scenario as seen from the frame of reference of the observer on the train.

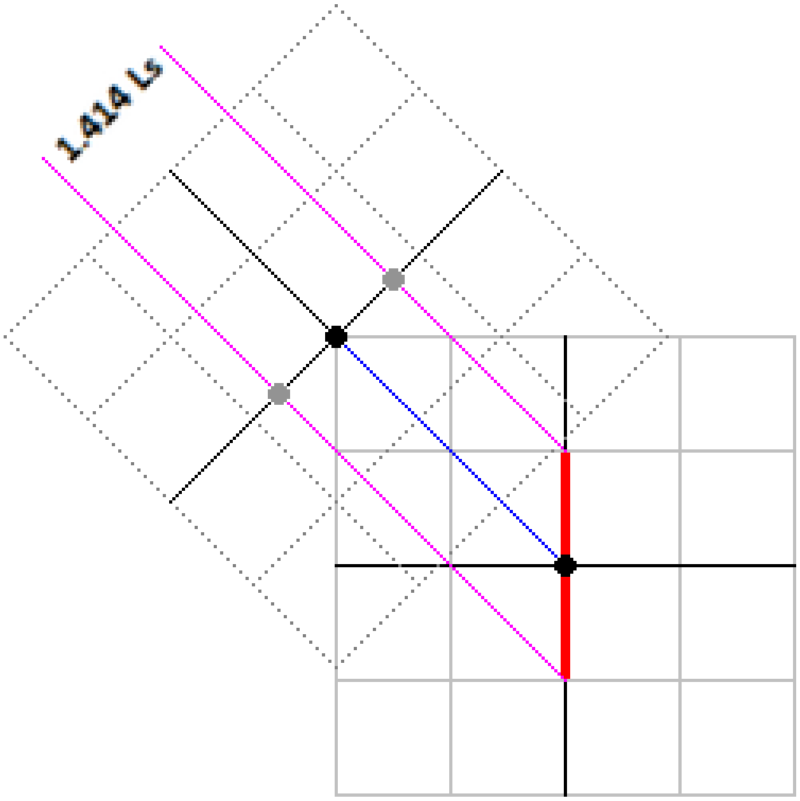
Above, we see the scenario two seconds before the observers meet, as seen by the observer on the train. This alignment of the frames of reference makes it easier to see that the observer by the track takes 2.83 seconds to travel two seconds in the frame of reference of the observer on the train. Again, each observer sees the other moving slower through time than him or herself.

Now, we have another problem. We tailored the experiment such that the two lightning bolts strike the opposite ends of the length-contracted train simultaneously, as seen by the observer on the track. As such, we put the lightning bolts 1.414 light seconds apart to match the length of the length-contracted moving train. The observer on the train does not see the train as length-contracted but sees it as its full two light seconds length. In addition, the observer on the train sees everything outside the train as moving; therefore, the space outside the train is length-contracted. This includes the space between the points in space where the lightning bolts strike. That space is length contracted such that the observer on the train sees them as only 0.9997 light seconds apart (1.414 Ls x 0.707), about half the length of the train. They are now too close together to reach both ends of the train simultaneously.



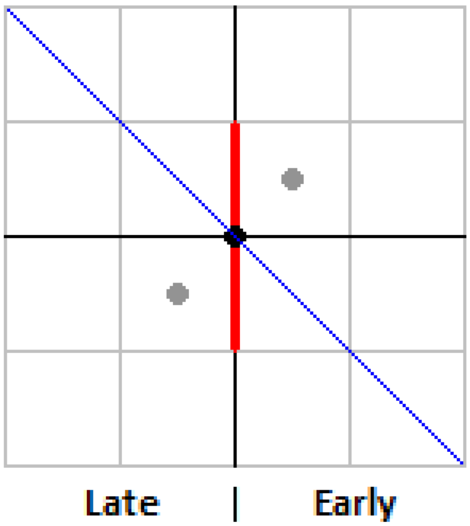
In the frame of reference of the observer on the train, the points in space where the lightning strikes are only 0.9997 Ls apart.

However, that is misleading. Changing our focus to the frame of reference of the observer on the train rotates the frames of reference to a new orientation, but the angles remain the same. The train still extends 1.414 Ls in the frame of reference of the observer by the track, and the points in space where the lightning strikes are still 1.414 Ls apart in that frame.



The length of the train still matches the distance between the lightning strikes in the frame of reference of the observer by the track.

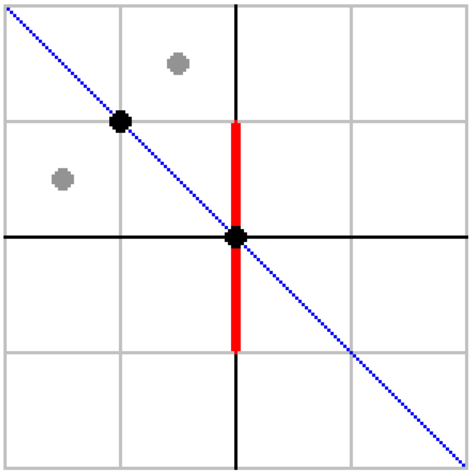
Now, let's move ahead to the point in time when the observers meet.



The front bolt struck 0.4998 seconds before the observers met, but the rear bolt will not strike until 0.4998 seconds after the observers met.

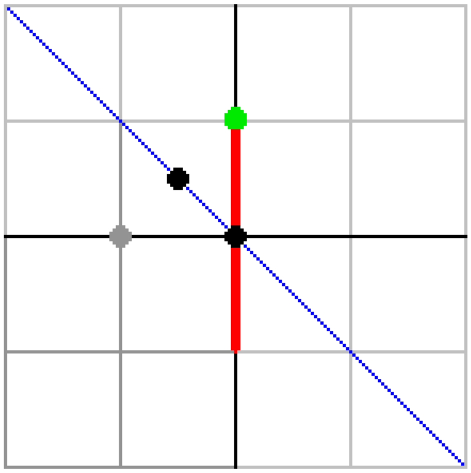
The lightning bolt that strikes the front of the train occurs 0.4998 seconds early. This means that it has already struck by the time the observes meet. Likewise, the bolt that strikes the rear of the train occurs 0.4998 seconds late. This means it won't strike until 0.4998 seconds after the observers meet. The observer by the track sees the lightning bolts strike simultaneously, but the observer on the train sees the front of the train struck 0.9997 seconds before the rear.

When focusing on the frame of reference of the observer on the train, we can look at the scenario as time passes and get a functional visual representation of the scenario over time. Let's start at one second before the observers meet in the frame of reference of the observer on the train.



The observers are one second from meeting.

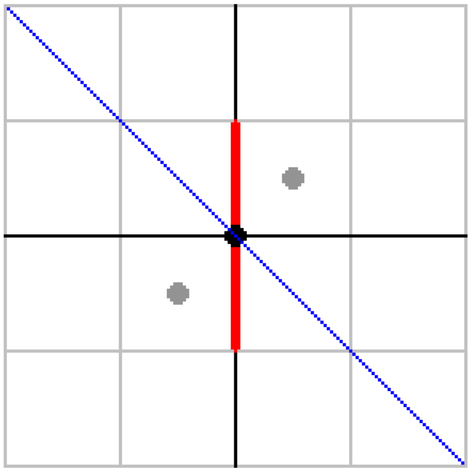
At one second before the observers meet (T-minus one second), the point in space where the rear bolt will strike is one-quarter of the way past the front of the train, and the observer by the track is just passing the front of the train.



0.4998 seconds before the observers meet

At 0.4998 seconds before the observers meet, the point in space where the front bolt strikes meets the front of the train. Recall that the graph indicates that the front lightning strike occurs 0.4998 seconds before the observers meet. Therefore, this is when the lightning strikes the front of the train. Conveniently, the point in space where the front lightning bolt strikes aligns with the front of the train on the graph at this time. Therefore, the observer on the train sees lightning strike the front of the train one-half second before the observer by the track arrives alongside the observer on the train.

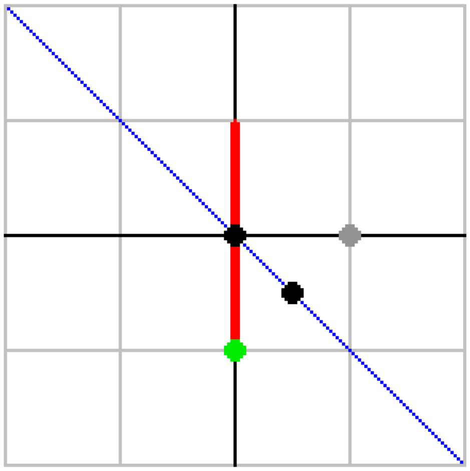
Now, let's look again at when the observers meet.



When the observers meet, the lightning has already struck the front of the train, as seen by the observer on the train.

When the observer by the track arrives alongside the observer on the train, the front bolt has already struck, but the rear bolt will not strike for another 0.4998 seconds.

Now, let's move to the time that the point in space where the rear bolt strikes aligns with the rear of the train.



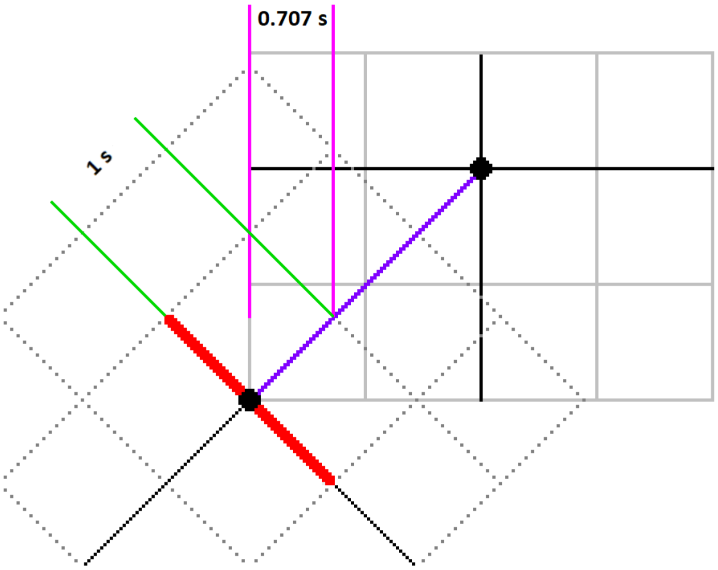
0.4998 seconds after the observers meet. The point in space and time when and where the rear bolt strikes meet.

At 0.4998 seconds after the observers meet, the point in space where the rear bolt strikes arrives at the rear of the train. The graph indicates that the bolt strikes 0.4998 seconds after the observers meet. Therefore, the bolt strikes at this time. Again, the point in space where the rear lightning bolt strikes conveniently lines up with the rear of the train on the graph at that time.

The observer on the train sees lightning strike the front of the train 0.9997 seconds before seeing it strike the rear of the train. The observer by the track sees lightning strike the ends of the train simultaneously, but the observer on the train sees the front get struck before the rear.

Reconciling time dilation

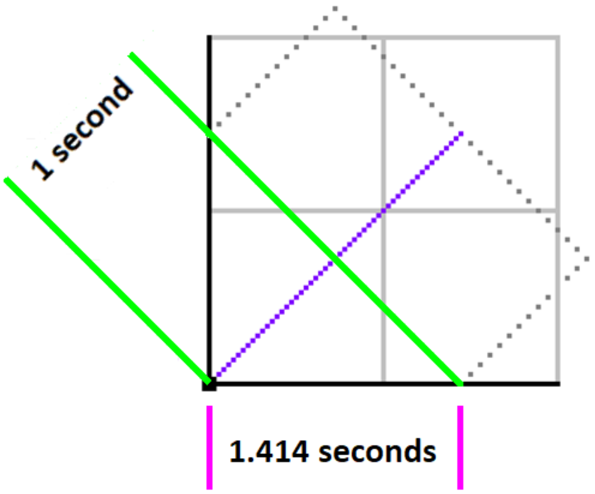
The graph initially gives a misleading visual representation of how space and time interact.



A clock possessed by the moving observer appears to tick off one second in only 0.707 seconds in the stationary observers' frame of reference. The moving clock is apparently ticking faster.

In the above illustration, we see that one second in the moving observer's frame of reference traverses only 0.707 seconds of the stationary observer's frame of reference. This makes it appear that the stationary observer would see a clock possessed by the moving observer as ticking off 2.83 seconds when his or her own clock had only ticked off two seconds; the moving clock ticking faster than the stationary clock.

To correctly interpret the graph, we must look at where timelines cross points in space. It is easiest to interpret how time and space interact if we look at time after the observers meet. Let's look at the point in time 1.414 seconds after the observers meet in the frame of reference of the observer by the track.



The moving observer's one-second timeline (green line) intersects the stationary observer's position in space (horizontal black line) at 1.414 seconds in the stationary observer's frame of reference.

The above illustration shows that the one-second timeline of the moving observer's frame of reference (green line) intersects the stationary observer's location in space (horizontal black line) at the stationary observer's 1.414-second mark. This means that one second in the moving observer's frame of reference corresponds to 1.414 seconds in the stationary observer's frame of reference; when the moving observer's clock ticks off one second, the stationary observer's clock has already gone through 1.414 seconds. The stationary observer sees the moving observer's clock ticking 29.3 percent slower than the stationary observer's clock. This shows that to "see" the other observer's clock, either observer must "look back" along the other observer's timeline to see what time the other observer's clock is showing.

Conclusion

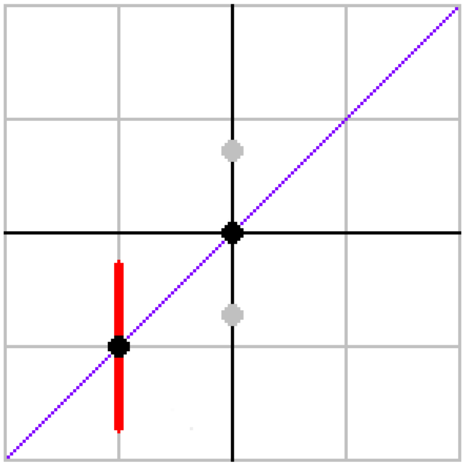
Einstein did not use Euclidean geography to develop his theory of special relativity. However, shortly after Einstein published his theory, Hermann Minkowsky developed a graphical representation of special relativity similar to the Euclidean geography above. Minkowsky's system is difficult for the uninitiated to understand. However, the above elementary geography is more accessible for an average educated person to visualize.

Addendum 1

If the graph is difficult to interpret when focusing on the frame of reference of the observer by the track, you're in luck. You can extrapolate it to a graph that gives a good visualization. All you have to do is rotate the train on an axis centered on the observer on the train—keeping the train length contracted—to be parallel with the stationary observer's timelines. This does not change when and where events occur but only changes our perspective. Now, we have a graph that can be animated to give a useful visual representation of the train moving through space and time.

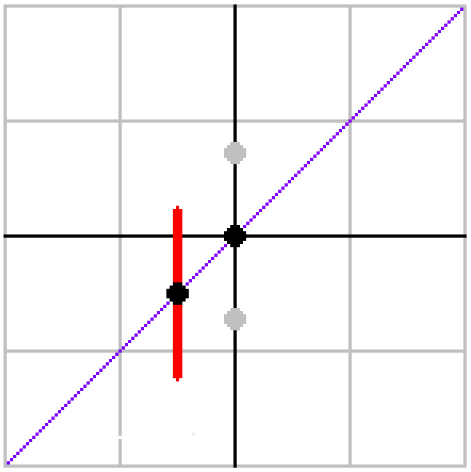
I should point out that as we look at the graph at different times, the observer by track should be on the same vertical line (at the same point in time) as the observer on the train; as time passes, the observer by the track should move along the graph to remain with the observer on the train along the vertical axis. If we were using a moving animation, this may make a good visualization. However, in this written treatise, we must use a few "freeze frames" instead. In this case, it may make a better visualization if we anchor the observer by the track at the point in time where he or she is when the lightning strikes.

Let's move the train back along the track and start it moving again toward the observer by the track at speed.



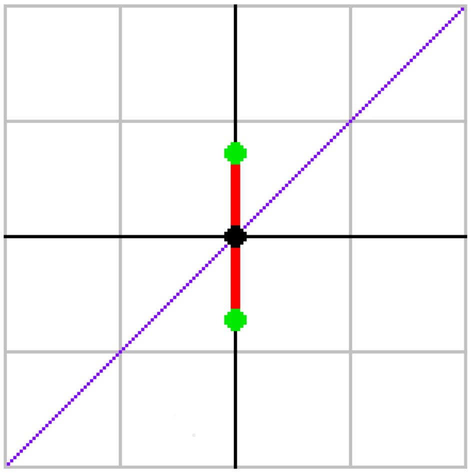
The train has been rotated to give a better visual representation of the train as it moves through space and time.

In the above diagram, the observer on the train is now one second (T-1 second) from meeting the observer by the track. The points in space where the lightning bolts strike are shown in gray because the lightning hasn't struck yet.



T-0.5s, one-half second before the lightning strikes.

As time passes, we approach the time that the lightning strikes.



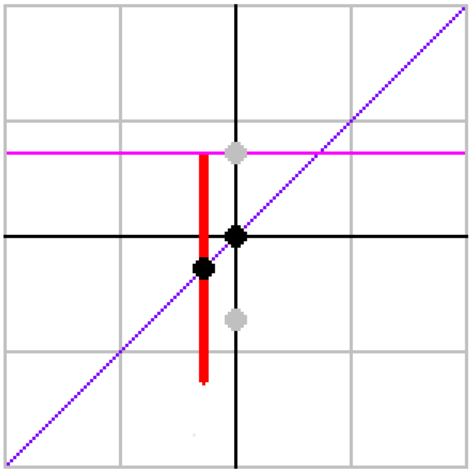
T=0, the lightning strikes simultaneously (green dots), as seen by the observer by the track.

Finally, we see the bolts of lightning, which the observer by the track sees simultaneously, striking the ends of the train just as it passes. After this, the train will continue its journey along the track.

Addendum 2

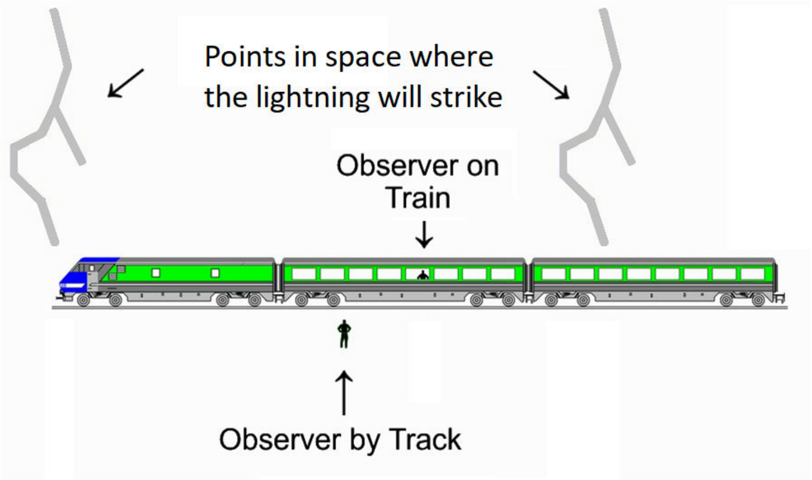
If the train were not length contracted, but the lightning strikes were 1,414 Ks apart, the front of the train would arrive at the point of the front lightning strike 0.293 seconds before the strike. The lightning would also strike 0.293 Ls behind the front of the train. However, since the front of the length-contracted train arrives at the point of the lightning strike 0.293 seconds later than the front of the full-length train, it arrives at the correct time to be struck. Likewise, the rear of the non-length-contracted train would arrive at the point of the rear lightning strike 0.293 seconds after the strike, and the lightning would strike 0.293 light seconds ahead of the rear of the train. But the rear of the length-contracted train arrives 0.293 seconds earlier than the rear of the full-length train at the correct time to be struck.

Let's see what would happen if the train were not length contracted.



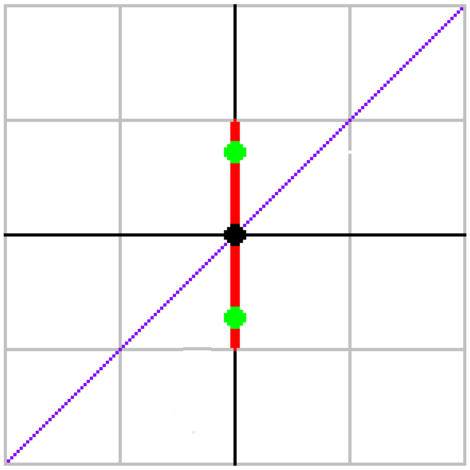
Full-length train at T-0.293 seconds

At T-0.293 seconds, the front of the full-length train has already reached the point in space where the front bolt will strike (the magenta line); the front of the full-length train is too early to be struck. This translates to a view from the perspective of the observer by the track as follows:



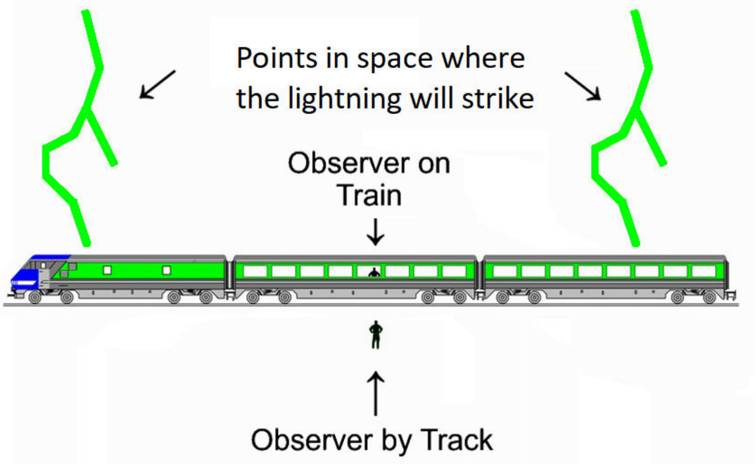
If the train were not length contracted, the front of the train would arrive at the point in space where the front lightning bolt strikes 0.293 seconds too early.

In that case, when the observer on the train reaches the observer by the track—the point in time the lightning strikes in the frame of reference of the observer by the track—the train will be struck, but the bolts will be too close together to reach the ends of the train.



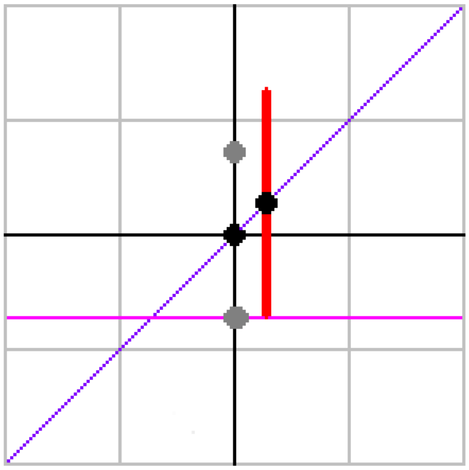
At T=0, the observer by the track sees the lightning strike. The full-length train is too long.

The graph shows this, and the observer would see it as below.



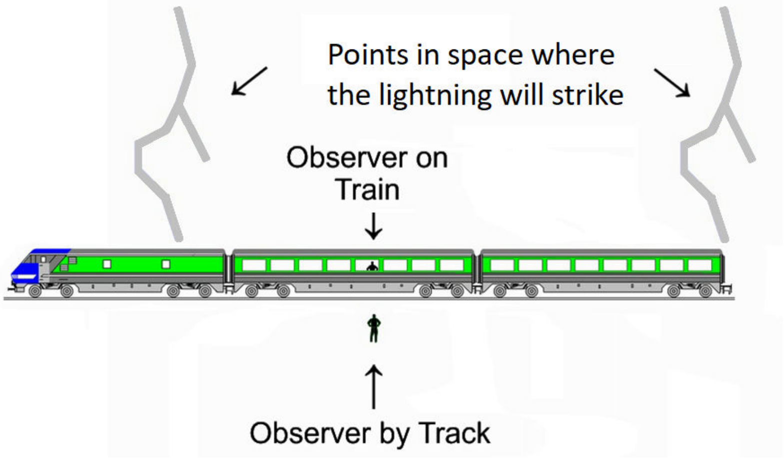
To the observer by the track, the full-length train is too long, and the lightning bolts don't strike the ends.

At T+0.293 seconds, the rear of the train reaches the point in space where the rear bolt strikes (magenta line), but is too late.



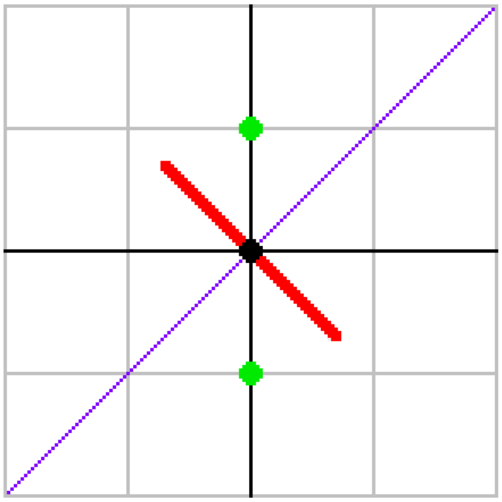
The observer by the track sees the rear of the train arrive 0.293 seconds after the lightning strikes.

The observer by the track would see it as below:



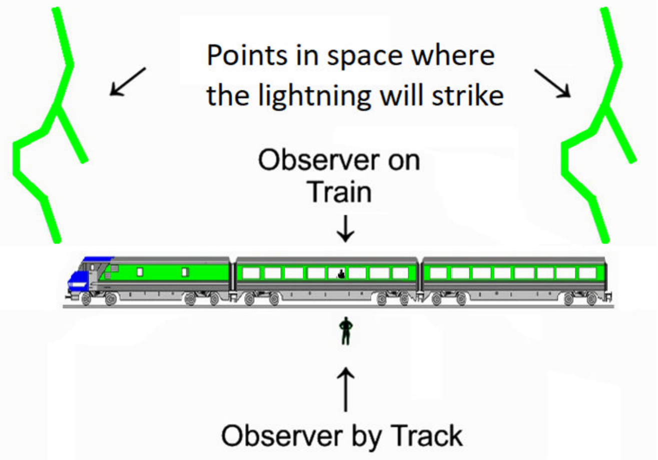
The rear of the full-length train reaches the point in space where the rear bolt strikes too late.

However, as the graph shows, the train is length contracted, causing the front of the train to arrive at the front strike point later and the rear of the train to arrive at the rear strike point earlier.



As described above, traveling at 70.7 percent of the speed of light, the train is rotated with the frame of reference of the observer on the train. This length contracts the train and causes the front to be early and the rear to be late in the frame of reference of the observer by the track.

This translates into the event as seen by the observer by the track as below:



The length contracted train is the correct length to be struck by the lightning bolts, and the ends of the length contracted train—the front being late and the rear being early—arrive at the points in space where the lightning strikes at the right time.